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A TERNARY SEARCH PROBLEM ON GRAPHS

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In a previous paper, Aigner studied the following search problem on graphs. For a graph G, let $e^* \in E(G)$ be an unknown edge. In order to find e^*, we choose a sequence of test-sets $A \subseteq V(G)$ where after every test we are told whether e^* has both end-vertices in A, one end-vertex, or none. Find the minimum $c(G)$ of tests required. Since in this problem ternary tests are performed, we have the usual information theoretic bound $\lceil \log_3 |E(G)| \rceil \leq c(G)$. Beside his main results which are on complete and complete bipartite graphs, Aigner proved that each forest F with maximum degree at most two is optimal, i.e., the information theoretic bound is achieved. In the present paper we consider the more general question, how close we can come to achieving the information theoretic bound for forests with maximum degree at most r, $r=1,2,\ldots$. Let $\mathscr{F}_r$ be the class of forests with non-empty edge-set and maximum degree at most r. We shall investigate the function $f(r)=\max\{c(F)-\lceil \log_3|E(F)|\rceil : F\in\mathscr{F}_r\}$ and obtain the result that $f(r)=t+1-\lceil \log_3(2^t+1)\rceil$ for $2^t<r\leq 2^{t+1}$, $t=0,1,\ldots$. In addition, we show that, with the exception of five small graphs, all members of $\mathscr{F}_3$ are optimal, and we conjecture that a similar result holds for $\mathscr{F}_r$, $r\geq 4$.

1. Introduction

Suppose that in a set S with n elements there are exactly d "defective" elements which are unknown. We wish to determine the set of defectives by a series of group tests. A test consists of selecting a subset A of S and, as the outcome of a test, one receives the number of defectives in A. Let $f_d(n)$ be the minimal number of tests sufficient to determine the defectives. It is well known that $f_1(n)=\lceil \log_2 n \rceil$, but for all $d\geq 2$ the question of determining the exact values of $f_d(n)$ is open and seems to be a difficult problem, even for $d=2$.

Let $d=2$ and assume that we have performed some tests on S. Then certain pairs are still candidates for being the defectives while others are already excluded, and thus we have a graph-structure on S. This observation leads to the following search problem for graphs which generalizes the above search problem for the case $d=2$.

Problem 1.1 (Aigner [2]). Given a finite simple graph G with vertex-set $V(G)$ and edge-set $E(G)$, and an unknown edge $e^* \in E(G)$. In order to find e^* we choose a sequence of test-sets $A \subseteq V(G)$ where after every test we are told whether e^* has both end-vertices in A, one end-vertex, or none. Find the minimum number $c(G)$ of tests required.

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Clearly, the original problem corresponds to the case that G is the complete graph K_n. For related problems, see the papers by Lindström [10–12], Cantor and Mills [5], Erdös and Rényi [8], Chang and Hwang [6], Chang, Hwang and Lin [7], Sobel [13], Tošić [14], Aigner and Schughart [3] and the bibliography of [2, 12]. For a general survey on search problems, see Katona [9] or the book of Ahlswede and Wegener [1]. We stress the fact that, in contrast to the papers [5, 8, 10–12], the selection of a test set may depend on the outcome of previous tests in the series, i.e., we do not restrict ourselves to predetermined strategies.

Since in the above problem we perform ternary tests, we have the usual information theoretic bound for $c(G)$:

$$c(G) \geq \lceil \log_3 q \rceil, \quad q = |E(G)|. \tag{1}$$

A graph G is *optimal* if equality holds in (1). In [2], Aigner received bounds on $c(K_n)$ and $c(K_{m,n})$ and determined the exact values for $c(K_{m,n})$ when $m=2,3,4$ (the case $m=1$ being trivial). In addition, it was proved in [2] that any forest with maximum degree at most two is optimal [2, Proposition 1] and some examples for non-optimal forests with maximum degree three and four were given.

Considering these results, one natural question to ask is "how close can we come to achieving the information theoretic bound (1) for forests with maximum degree at most r?" In the present paper, we give the following answer to this question. For a graph G, let $\Delta(G)$ denote its maximum degree and let $\mathscr{F}_r$ be the class of forests F with $1 \leq \Delta(F) \leq r$, $r=1,2,\ldots$. We shall study the function

$$f(r) = \max\{c(F) - \lceil \log_3 |E(F)| \rceil : F \in \mathscr{F}_r\}$$

and obtain the result (Theorem 3.2) that

$$f(r) = t+1-\lceil \log_3(2^t+1) \rceil \qquad \text{for } 2^t < r \leq 2^{t+1}, \quad t = 0, 1, \ldots.$$

In particular, it follows that $f(r) \sim (1-\log_3 2)\log_2 r$. Table 1 displays values of $f(r)$ for $r \leq 2^{14}$.

Table 1.

	$1\leq r\leq 2$	$3\leq r\leq 8$	$9\leq r\leq 64$	$65\leq r\leq 512$	$513\leq r\leq 2048$	$2049\leq r\leq 16384$
$f(r)$	0	1	2	3	4	5

In [2] the problem was posed to characterize all optimal forests. As a partial result on this problem, we prove that all members of $\mathscr{F}_3$ are optimal, except for five forests which are shown in Fig. 1 (Theorem 4.1). We conjecture that a similar result holds for all positive integers r.

Conjecture 1.2. *With only a finite number of exceptions, all members of $\mathscr{F}_r$ are optimal.*

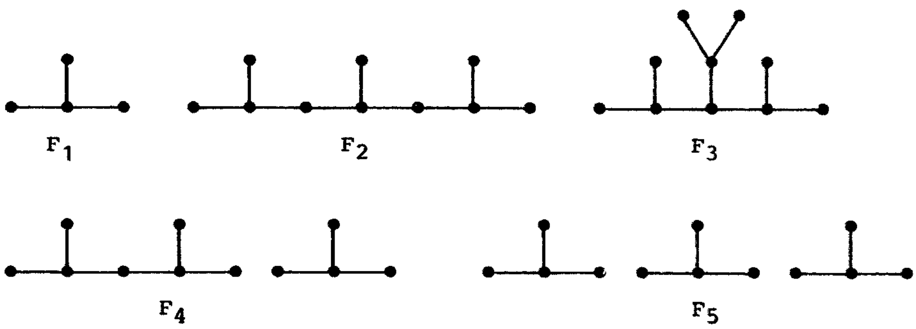

Fig. 1.

The proofs of Theorem 3.2 and 4.1 are based on the following result (Theorem 2.2). Suppose that the vertices of a graph G are colored with two colors red an blue. Call an edge of G *red* if it connects two red vertices; similarly a *blue* (*mixed*) edge is defined. Call G 3-*divisible* if it is possible to 2-color the vertices of G with red and blue such that the number of red, blue and mixed edges each is exactly $\frac{1}{3}q$, where $q=|E(G)|$. If a graph G with q edges and $\Delta(G)=r$ is 3-divisible, then clearly $q\equiv 0(\text{mod } 3)$ and $q\geq\frac{3}{2}r$. Theorem 2.2 states that this necessary condition for 3-divisibility is also sufficient when G is a forest.

For graph-theoretical terminology not defined in this paper, we refer to Bondy and Murty [4]. The degree of a vertex v in a graph G is denoted $d(v,G)$. Any time we 2-color the vertices of a graph, we assume without any further mentioning that the colors used are red an blue. A *proper* 2-*coloring* of the vertices of a graph G is one that results in the same number of red, blue and mixed edges, namely $\frac{1}{3}q$ where $q=|E(G)|$. $\mathbb{N}$ denotes the positive integers.

2. 3-divisible forests

For the proof of Theorem 2.2 we need a lemma which also will be used in Section 4.

Lemma 2.1. *For each tree T with q edges one of the following holds.*

(a) *There is an edge e such that each component of $T-e$ has at least $\frac{1}{3}q$ edges.*

(b) *There is a vertex v such that each component of $T-v$ has fewer than $\frac{1}{3}q$ edges.*

Proof. For $e\in E(T)$, let $h(e)=\min\{|E(C)|: C \text{ component of } T-e\}$ and pick an edge e_0 such that $h(e_0)$ is maximal. Let A and B be the components of $T-e_0$, where $h(e_0)=|E(A)|$ and let v be the vertex of B which is incident with e_0. If $h(e_0)\geq\frac{1}{3}q$, then (a). Thus assume that $h(e_0)<\frac{1}{3}q$. If $E(B)=\varnothing$, then $E(T)=\{e_0\}$

and (b) clearly holds. If $E(B) \neq \emptyset$, then let C be a component of $T-v$, $C \neq A$, and let e_1 be the edge that joins v with C. Let A^+ be the component of $T-e_1$ that contains v. Then

$$\min\{|E(A^+)|, |E(C)|\} = h(e_1) \leq h(e_0) = |E(A)| < |E(A^+)|$$

and thus

$$|E(C)| = h(e_1) \leq h(e_0) < \tfrac{1}{3}q.$$

Hence (b). □

Let G be a 3-divisible graph with q edges and $\Delta(G) = r$. Then $q \equiv 0 (\mathrm{mod}\ 3)$; further, let v be a vertex of degree r and, for a proper 2-coloring of G, let v be colored, say, red. Then the edges incident with v are either red or mixed and thus, since there are $\frac{1}{3}q$ blue edges, $r + \frac{1}{3}q \leq q$. Hence $q \geq \frac{3}{2}r$. In general, the conditions $q \equiv 0 (\mathrm{mod}\ 3)$ and $q \geq \frac{3}{2}r$ are not sufficient for 3-divisibility of a graph as can be seen by considering a triangle or $K_{2,3}$, however, for forests we have the following result.

Theorem 2.2. *A forest F with q edges and $\Delta(F) = r$ is 3-divisible if and only if $q \equiv 0 (\mathrm{mod}\ 3)$ and $q \geq \frac{3}{2}r$.*

Proof. By the above remark, it remains to establish sufficiency. Hence let $q \equiv 0 (\mathrm{mod}\ 3)$ and $q \geq \frac{3}{2}r$. The case $q = 0$ is trivial and so let $q \geq 3$. If F is disconnected, then let $F_1, \ldots, F_t$ be the nontrivial components of F. For each i, pick distinct end-vertices x_i, y_i of F_i and let F' be the tree that results by identifying y_i with x_{i+1}, $i = 1, \ldots, t-1$. Then

$$|E(F')| = q, \qquad \Delta(F') = \max\{2, r\}$$

and thus $|E(F')| \geq \frac{3}{2}\Delta(F')$. Moreover, when F' is 3-divisible then the same holds for F. From this one concludes that it is sufficient to prove our theorem for trees, and thus let us assume that F is a tree.

Suppose that there is an edge e such that each component of $F-e$ has at least $\frac{1}{3}q$ edges. Let A and B be the components of $F-e$ and let $e = (a, b)$, $a \in A$, $b \in B$. Because $|E(A)|, |E(B)| \geq \frac{1}{3}q$, we can pick two subtrees A' and B' of F such that $a \in A' \subseteq A$, $b \in B' \subseteq B$, $|E(A')| = |E(B')| = \frac{1}{3}q$. Color the vertices of A' (B') red (blue). Then e is mixed and one easily finds that one can color the remaining vertices such that all edges outside $A' \cup B'$ are mixed. Thus F is 3-divisible.

Hence by the lemma we may assume that there is a vertex v such that, for the components $A_1, \ldots, A_t$ of $F-v$, we have

$$\tfrac{1}{3}q > |E(A_1)| \geq |E(A_2)| \geq \cdots \geq |E(A_t)|. \tag{2}$$

Assume that $\sum_{i=1}^{t} |E(A_i)| < \frac{1}{3}q$. Then, because $t + \sum_{i=1}^{t} |E(A_i)| = q$, it would follow that $t > \frac{2}{3}q$ which implies $r \geq d(v, F) = t > \frac{2}{3}q$, contradicting the hypothesis that $q \geq \frac{3}{2}r$. Hence $\sum_{i=1}^{t} |E(A_i)| \geq \frac{1}{3}q$ and thus we may define s as the least number for which $\sum_{i=1}^{s} |E(A_i)| \geq \frac{1}{3}q$. We claim that

$$s + \sum_{i=1}^{s} |E(A_i)| \le \tfrac{2}{3}q. \tag{3}$$

If $|E(A_s)| \le 1$, then it follows from the minimal choice of s that

$$\sum_{i=1}^{s} |E(A_i)| = \tfrac{1}{3}q \quad \text{and} \quad |E(A_s)| = 1,$$

and thus by (2), $s \le \sum_{i=1}^{s} |E(A_i)| = \frac{1}{3}q$ which implies (3). If $|E(A_s)| \ge 2$, then it follows from (2) and the minimal choice of s that

$$2(s-1) \le \sum_{i=1}^{s-1} |E(A_i)| \le \tfrac{1}{3}q - 1.$$

From this one easily finds that

$$s + \sum_{i=1}^{s-1} |E(A_i)| \le \tfrac{1}{2}(q-1).$$

Moreover by (2), we have

$$s \ge 2 \quad \text{and} \quad 2 + |E(A_1)| + |E(A_2)| \le \tfrac{2}{3}q,$$

and so we may assume that $s \ge 3$. It follows that $|E(A_s)| < \frac{1}{6}q$ since, otherwise,

$$|E(A_1)| + |E(A_2)| \ge 2|E(A_s)| \ge \tfrac{1}{3}q$$

which would imply $s \le 2$. Hence

$$s + \sum_{i=1}^{s} |E(A_i)| \le \tfrac{1}{2}(q-1) + \tfrac{1}{6}q < \tfrac{2}{3}q.$$

Let T be the subtree of F which is spanned by $\bigcup_{i=s+1}^{t} V(A_i) \cup \{v\}$. It follows from (3) that $|E(T)| \ge \frac{1}{3}q$. Thus we can find a subtree T' of T with $|E(T')| = \frac{1}{3}q$ and $v \in T'$. Let a_i be the vertex of A_i which is adjacent to v, $i = 1, \ldots, s$. By the choice of s, there exists a subtree A_s' of A_s such that

$$a_s \in A_s' \quad \text{and} \quad \sum_{i=1}^{s-1} |E(A_i)| + |E(A_s')| = \tfrac{1}{3}q.$$

Let

$$H = A_1 \cup \cdots \cup A_{s-1} \cup A_s'.$$

We now color the vertices of T' red and the vertices of H blue. Then we obtain $\frac{1}{3}q$ red edges and $\frac{1}{3}q$ blue edges and the edges (v, a_i), $i = 1, \ldots, s$, are mixed. Clearly one can color the remaining vertices such that all other edges are mixed. □

3. Forests with maximum degree at most r

We shall frequently make use of the following straightforward connection between 2-colorings and our search problem. As before, let $e^* \in E(G)$ be the unknown edge. Suppose that G is 2-colored and let G_1, G_2 and G_3 be the resulting graphs induced by the red and blue and mixed edges, respectively. If we use the set of red vertices as our first test set, then this test will tell us whether e^* is red, blue or mixed, and we can continue with a test on the graph G_i which contains e^*. Hence $c(G) \leq 1 + \max\{c(G_i)\colon 1 \leq i \leq 3,\ E(G_i) \neq \emptyset\}$.

As an immediate consequence of [2, Proposition 2] we have

$$c(F) \leq \lceil \log_2 |E(F)| \rceil \tag{4}$$

for each forest F with $E(F) \neq \emptyset$. (Alternatively, (4) can also be proved without referring to [2] by making use of the fact that each tree with q edges contains a subtree with $\lceil \frac{1}{2}q \rceil$ edges.)

Proposition 3.1. *Let F be a forest with q edges and $\Delta(F) = r$ and suppose that $1 \leq q \leq \frac{1}{2}r$. Then $c(F) \leq \lceil \log_2 r \rceil$.*

Proof. This is trivial for $q = 1$, so let $q \geq 2$. Let v be a vertex with $d(v, F) = r$. Because $q \leq \frac{1}{2}r$ one can find $l := \lceil r/2 \rceil$ neighbors $v_1, \ldots, v_l$ of v which are end-vertices of F. Color $v, v_1, \ldots, v_l$ red and all other vertices blue and let L_1, L_2 and L_3 be the subgraphs of F induced by the red and blue and mixed edges, respectively. Then $|E(L_i)| \leq l$, $i = 1, 2, 3$, and thus

$$c(F) \leq 1 + \max\{c(L_i)\colon 1 \leq i \leq 3,\ E(L_i) \neq \emptyset\} \leq 1 + \lceil \log_2 l \rceil,$$

where the last inequality follows from (4). Because $2 \leq q \leq \frac{1}{2}r$ we have $r \geq 2$ and thus $1 + \lceil \log_2 l \rceil = \lceil \log_2 r \rceil$. □

Theorem 3.2. *Let*

$$f(r) = \max\{c(F) - \lceil \log_3 |E(F)| \rceil\colon F \textit{ forest with } 1 \leq \Delta(F) \leq r\},$$

$r = 1, 2, \ldots$. *Then $f(1) = 0$ and $f(r) = t + 1 - \lceil \log_3(2^t + 1) \rceil$ for $2^t < r \leq 2^{t+1}$, $t = 0, 1, \ldots$.*

Proof. Let $g(1) = 0$ and $g(r) = t + 1 - \lceil \log_3(2^t + 1) \rceil$ for $2^t < r \leq 2^{t+1}$, $t = 0, 1, \ldots$. We show that $c(F) \leq \lceil \log_3 q \rceil + g(r)$ for a forest F with q edges and $\Delta(F) = r$. First we consider the case that $q = 3^k$ for some $k \in \{0, 1, \ldots\}$. We proceed by induction on k. For $k = 0$, the assertion is trivial. Let $k \geq 1$. If $q < \frac{1}{2}r$, then $c(F) \leq \lceil \log_2 r \rceil$ by Proposition 3.1. Note that $r > 1$ since $3 \leq q < \frac{1}{2}r$. As above let $t = \lceil \log_2 r \rceil - 1$. Then $\frac{1}{2}r < 2^t + 1 \leq r \leq q = 3^k < \frac{3}{2}r$ and thus $\lceil \log_3(2^t + 1) \rceil = k$. Hence $\lceil \log_3 q \rceil + g(r) = \lceil \log_2 r \rceil \geq c(F)$. If $q \geq \frac{3}{2}r$, then by Theorem 2.2 we can 2-color the vertices of F such that the three corresponding graphs induced by the red, blue or mixed edges each have 3^{k-1} edges. Choose F' as one of these graphs such that $c(F')$ is maximal. Let $r' = \Delta(F')$; we have $r' \leq r$ and thus $g(r') \leq g(r)$. Hence by induction

$$c(F) \leq c(F') + 1 \leq k - 1 + g(r') + 1 \leq k + g(r) = \lceil \log_3 q \rceil + g(r).$$

If $3^{k-1} < q < 3^k$, then we add $3^k - q$ isolated edges to F and consider the resulting graph F^+. Then

$$c(F) \leq c(F^+) \leq k + g(r) = \lceil \log_3 q \rceil + g(r).$$

Since $r' \leq r$ implies $g(r') \leq g(r)$, it follows that $f(r) \leq g(r)$, $r = 1, 2, \ldots$. Since $g(1) = 0$, we have $f(1) = g(1)$. Let $2^t < r \leq 2^{t+1}$ for $t \in \{0, 1, \ldots\}$ and let F be the star $K_{1,m}$ where $m = 2^t + 1$. Clearly $c(F) = \lceil \log_2 m \rceil$ and thus

$$f(r) \geq c(F) - \lceil \log_3 |E(F)| \rceil = \lceil \log_2 m \rceil - \lceil \log_3 m \rceil = g(m) = g(r). \qquad \square$$

4. Forests with maximum degree at most three

The following theorem provides a partial solution to a problem posed by Aigner [2]. As mentioned above, we conjecture that a similar result holds for the general case of forests with maximum degree at most r.

Theorem 4.1. *Let F be a forest with $E(F) \neq \varnothing$ and $\Delta(F) \leq 3$ and assume that F is not isomorphic to one of the forests F_i, $i = 1, \ldots, 5$, shown in Fig. 1. Then F is optimal.*

Proof. The case $|E(F)| \leq 27$ will be settled further down. Assume that we have already proved the theorem for forests with 3^{k-1} edges and let

$$3^{k-1} < |E(F)| \leq 3^k, \quad k \geq 4.$$

Add $3^k - |E(F)|$ isolated edges to F and call the resulting graph F^+. By Theorem 2.2, F^+ is 3-divisible; further, by the induction hypothesis, all subforests of F^+ with 3^{k-1} edges are optimal and thus also F^+ is optimal. Hence F is optimal and it remains to consider the case $|E(F)| \leq 27$. For forests F with a most three edges, our theorem clearly holds. In order to prove our theorem for forests with at most nine edges it suffices to consider the case that $|E(F)| = 9$, $F \not\cong F_i$, $i = 2, \ldots, 5$, which we assume now. We have to show that the vertices of F can be 2-colored such that each of the three resulting subgraphs is a forest G with $|E(G)| = 3$ and $G \not\cong F_1$. We claim that without loss of generality, we may assume that F is a tree. For the proof assume that F is disconnected and let the tree F' result from F as described in the proof of Theorem 2.2. Then $F' \not\cong F_3$ and since $F \not\cong F_4, F_5$ we also have $F' \not\cong F_2$. Moreover, optimality of F' clearly implies optimality of F and thus we may assume that F is a tree.

Case 1. There exists an edge e of F such that the components A and B of $F - e$ both have at least three edges. Let $e = (a, b)$, $a \in A$, $b \in B$ and pick two subtrees A' and B' of F such that $a \in A' \subseteq A$, $b \in B' \subseteq B$, $|E(A')| = |E(B')| = 3$, and as in the proof of Theorem 2.2 color the vertices of F such that the edges of A' (B') are red (blue) and the edges outside $A' \cup B'$ are mixed. Then the mixed graph is not isomorphic to F_1. If $A' \not\cong F_1 \not\cong B'$ then we are done, and thus we may assume that $A' \cong F_1$. Further, if A' is properly contained in A, then by an appropriate choice of A' it can be achieved that $A' \not\cong F_1$ and thus $A = A'$ may be assumed. If $d(b, F) = 2$, then the assertion follows by considering the edge e' instead of e, where e' is incident with b, $e' \neq e$. Hence $d(b,F)=3$. We now choose a new 2-coloring with exactly four red vertices which form a path of length three that connects b with an end-vertex of A. Then, because $F \not\cong F_2$, this coloring has the desired properties, i.e., each of the three resulting subgraphs G is a forest with $|E(G)|=3$ and $G \not\cong F_1$.

If there is no edge satisfying the hypothesis of Case 1, then by the lemma we are in the following case.

Case 2. There is a vertex $v \in F$ with $d(v,F)=3$ such that $|E(A_i)|=2$ for each of the components A_i of $F-v$, $i=1,2,3$. Since $F \not\cong F_3$, we may assume without loss of generality that the subtree spanned by $V(A_1) \cup \{v\}$ is not isomorphic to F_1. Let a_i be the neighbor of v which is contained in A_i, $i=1,2,3$ and let $x \neq v$ be a neighbor of a_3. Then we are done if we color a_3, x and the vertices of A_2 blue and the other vertices red.

It remains to consider the case $|E(F)|=27$. We have to show that F can be 2-colored such that each of the three resulting subgraphs is a forest with nine edges and that none of these subgraphs is isomorphic to one of the graphs F_i, $i=2,\ldots,5$. As above we may assume that F is a tree. The following proof will be organized similarly to the above proof for the case $|E(F)|=9$.

Let us first assume that there exists an edge e of F such that the components A and B of $F-e$ both have at least nine edges. Choose A', B', a, b similar as in the above Case 1 and, also as above, color the vertices of F such that the edges of A' (B') are red (blue) and the edges outside $A' \cup B'$ are mixed. Then the mixed graph is not isomorphic to one of the graphs F_i, $i=2,\ldots,5$, since each of a and b has a degree at most two in the mixed graph. Similar as in Case 1, we may assume that $A' \cong F_i$ for $i=2$ or 3. Then there are exactly four possible types for the rooted graph A' (with root a); these are shown in Fig. 2.

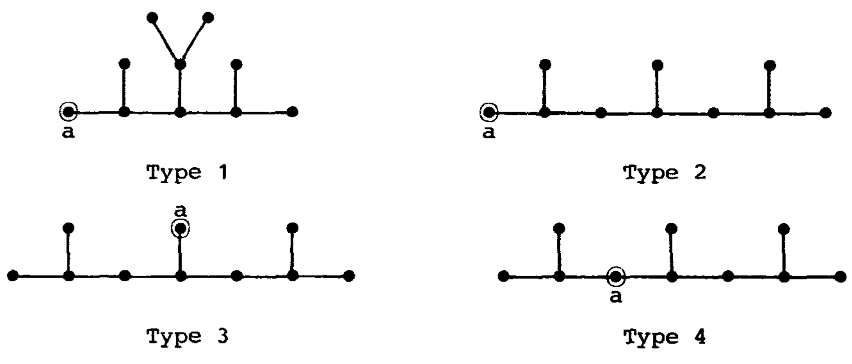

Fig. 2.

If A' is properly contained in A, then pick an edge $(x,y) \in E(A)$ such that $x \in A'$, $y \notin A'$, Let further y' be an end-vertex of A' such that $y' \neq a,x$. If we change the color of y into red and the color of y' into blue, then a look at Fig. 2 tells us that the resulting red tree is no longer isomorphic to F_2 or F_3. Thus we may assume that $A'=A$.

Let z be an end-vertex of F, $z\in A$. Let us change the color of z into blue and the color of b into red. As above the resulting red tree is not isomorphic to F_2 or F_3. If $d(b,F)=2$, then we are done since we may consider the edge e' instead of e, where e' is incident with b, $e'\neq e$. Hence $d(b,F)=3$. Let B_1 and B_2 be the components of $F-b$ which are distinct from A. Let b_i be the neighbor of b which is contained in B_i, $i=1,2$. If $E(B_i)=\emptyset$ for $i=1$ or 2, then we are done since we may consider $e'=(b,b_j)$ instead of e, where $j\neq i$. Otherwise, we can find a subforest F' of $B_1\cup B_2$ such that $b_1,b_2\in F'$ and such that F' consists of two components one of which has exactly eight edges and the other one is an isolated edge. In particular, $F'\not\cong F_i$, $i=2,\dots,5$.

Now, recolor the vertices of $B_1\cup B_2$ such that the edges of F' are blue and all other edges of $B_1\cup B_2$ are mixed. One easily finds that this yields a coloring of F as desired.

By the lemma, it remains to consider the case that there is a vertex $a\in F$ with $d(a,F)=3$ such that $|E(A_i)|=8$ for each of the components A_i of $F-a$, $i=1,2,3$. Let a_i be the neighbor of a which is contained in A_i and let A_i^+ be the graph spanned by $V(A_i)\cup\{a\}$, $i=1,2,3$. If one of A_i^+, say A_1^+, is not isomorphic to F_2 or F_3, then we are easily done: we just pick a subforest H of $A_2\cup A_3$ such that H consists of two components one of which is A_2 and the other is an isolated edge which is incident with a_3, and color the vertices of F such that exactly the edges of $A_1^+(H)$ are red (blue). Then all three resulting graphs are as desired. If all A_i^+ are isomorphic to F_2 or F_3, then we pick H in the same way as before and color the vertices of F as before, however, this time we take care that A_3 contains a mixed end-edge of F which has a blue end-vertex x. (Checking the three possible cases that A_3^+ is of type 1, 2 or 3 of Fig. 2, one easily finds that this is possible.) Let $y\in A_1$ be an endvertex of F. Changing the color of y into blue and the color of x into red, we get a coloring of F as desired. $\square$

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